

# Interval State Observer for Nonlinear Time Varying Systems

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## Abstract

This paper is devoted to design of interval observers for Linear Time Varying (LTV) systems and a class of nonlinear time-varying systems in the output canonical form. An interval observer design is feasible if it is possible to calculate the observer gains making the estimation error dynamics cooperative and stable. It is shown that under some mild conditions the cooperativity of an LTV system can be ensured by a static linear transformation of coordinates. The efficiency of the proposed approach is demonstrated through numerical simulations.

## I. INTRODUCTION

The problem of unmeasurable state vector estimation is very challenging and its solution is demanded in many applications [4], [5], [15]. In some situations due to presence of uncertainty (parametric or/and signal) the design of a conventional pointwise estimator, converging in the noise-free case to the ideal value of the state, is not possible. However, an interval estimation remains feasible. By interval or set-membership estimation we understand an observer that, using input-output information, computes an outer-approximation of the set of admissible values (interval) for the state at each instant of time. Another group of applications deals directly with evaluation of the set of admissible values for the state of an uncertain system (it can be the estimation goal in some fault detection systems, in biology or chemistry), the interval observers were proposed as a solution of this problem [6].

There are several approaches to design interval observers [3], [7], [8], [12]. This paper continues the framework of interval observer design based on the monotone system theory [3], [12]. That approach has been recently extended in [17] to nonlinear systems using Linear-Parameter-Varying (LPV) representation with known minorant and majorant matrices, and in [16] for observable nonlinear systems. One of the most complex assumptions for the interval observer design, dealing with cooperativity of the interval estimation error dynamics, was relaxed in [10], [16]. It was shown that under some mild conditions applying similarity transformation, a Hurwitz matrix could be transformed to Hurwitz and Metzler (cooperative). The transformation matrix is a solution of the Sylvester equation, a constructive procedure for this solution calculation was also given in [16].

The objective of this work is to develop the approach of interval observer design to the systems with non-constant matrices dependent on measurable input-output signals and time. In order to solve this problem an extension of the result from [16] is presented, that allows us to calculate a constant similarity transformation matrix representing a given interval of matrices to an interval of Metzler matrices. This result can be used to design interval observers for LPV systems with measurable vector of scheduling parameters [11], [18], [20] or LTV systems, that is the

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main novelty of the work. Two examples of such systems are considered in this work: LTV system and the Lorenz chaotic model (as a nonlinear system in the output canonical form).

The paper is organized as follows. Some basic facts from the theory of interval estimation are given in Section 2. The main result is described in Section 3. The examples of computer simulation are presented in Section 4.

## II. PRELIMINARIES

Euclidean norm for a vector  $x \in \mathbb{R}^n$  will be denoted as  $|x|$ , and for a measurable and locally essentially bounded input  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  ( $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ ) the symbol  $\|u\|_{[t_0, t_1]}$  denotes its  $L_\infty$  norm:

$$\|u\|_{[t_0, t_1]} = \text{ess sup}\{|u(t)|, t \in [t_0, t_1]\},$$

if  $t_1 = +\infty$  then we will simply write  $\|u\|$ . We will denote as  $\mathcal{L}_\infty$  the set of all inputs  $u$  with the property  $\|u\| < \infty$ . Denote the sequence of integers  $1, \dots, k$  as  $\overline{1, k}$ . The symbols  $I_n$  and  $E_n$  denote the identity matrix and the matrix with all elements equal 1 respectively (with dimension  $n \times n$ ). For a matrix  $A \in \mathbb{R}^{n \times n}$  the vector of its eigenvalues is denoted as  $\lambda(A)$ ,  $\|A\|_{max} = \max_{i=\overline{1, n}, j=\overline{1, n}} |A_{i,j}|$  (the elementwise maximum norm, it is not sub-multiplicative) and  $\|A\|_2 = \sqrt{\max_{i=\overline{1, n}} \lambda_i(A^T A)}$  (the induced  $L_2$  matrix norm), the relation

$$\|A\|_{max} \leq \|A\|_2 \leq n\|A\|_{max} \quad (1)$$

is satisfied between these norms.

For two vectors  $x_1, x_2 \in \mathbb{R}^n$  or matrices  $A_1, A_2 \in \mathbb{R}^{n \times n}$ , the relations  $x_1 \leq x_2$  and  $A_1 \leq A_2$  are understood elementwise. The relation  $P \succ 0$  means that the matrix  $P \in \mathbb{R}^{m \times n}$  is positive definite. Given a matrix  $A \in \mathbb{R}^{m \times n}$  define  $A^+ = \max\{0, A\}$ ,  $A^- = A^+ - A$ .

**Lemma 1.** *Let  $x \in \mathbb{R}^n$  be a vector variable,  $\underline{x} \leq x \leq \bar{x}$  for some  $\underline{x}, \bar{x} \in \mathbb{R}^n$ , and  $A \in \mathbb{R}^{m \times n}$  be a constant matrix, then*

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}. \quad (2)$$

*Proof:* Note that  $Ax = (A^+ - A^-)x$ , that for  $\underline{x} \leq x \leq \bar{x}$  gives the required estimates. ■

A matrix  $A \in \mathbb{R}^{n \times n}$  is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\dot{x} = Ax + \omega(t), \quad \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^n,$$

with  $x \in \mathbb{R}^n$  and a Metzler matrix  $A$ , is elementwise nonnegative for all  $t \geq 0$  provided that  $x(0) \geq 0$  [19]. Such dynamical systems are called cooperative (monotone) [19].

**Lemma 2.** [16] *Given the matrices  $A \in \mathbb{R}^{n \times n}$ ,  $R \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{p \times n}$ . If there is a matrix  $L \in \mathbb{R}^{n \times p}$  such that the matrices  $A - LC$  and  $R$  have the same eigenvalues, then there is a  $S \in \mathbb{R}^{n \times n}$  such that  $R = S^{-1}(A - LC)S$  provided that the pairs  $(A - LC, e_1)$  and  $(R, e_2)$  are observable for some  $e_1 \in \mathbb{R}^{1 \times n}$ ,  $e_2 \in \mathbb{R}^{1 \times n}$ .*

This result was used in [16] to design interval observers for linear time invariant systems with a Metzler matrix  $R$  (the main difficulty is to prove the existence of a *real* matrix  $S$ , and to provide a constructive approach of its calculation).

### III. MAIN RESULT

In this work we consider the following model of a nonlinear time-varying system:

$$\begin{aligned}\dot{x} &= A(t, y, u)x + f(t, x, u, \varrho), \\ y &= C(t, u)x,\end{aligned}\tag{3}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^p$  are the state, the input and the output of the system (3),  $\varrho \in \Theta \subset \mathbb{R}^q$  is the vector of unknown signals or parameters, the compact set  $\Theta$  is given, the matrix functions  $A : \mathbb{R}^{p+m+1} \rightarrow \mathbb{R}^{n \times n}$ ,  $C : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{p \times n}$  and the function  $f : \mathbb{R}^{n+m+q+1} \rightarrow \mathbb{R}^{n \times m}$  are given. The instant values of  $u(t) \in \mathcal{L}_\infty$ ,  $y(t) \in \mathcal{L}_\infty$  are known. In this work we consider the case without measurement noise, the proposed result can be extended to the case with a noise in the measurement channel, this extension is omitted for brevity of presentation. Denoting  $\theta(t) = [t \ y \ u]^T$  we can rewrite the system (3) in the quasi-LPV form with a measurable scheduling parameter vector  $\theta$ .

Many works on the interval observer design [3], [12], [17], [16] deal with the case of a constant matrix  $A$  (or under some transformations the estimation error can be represented in the form with a constant matrix  $A$ , next an observer gain  $L$  can be found such that  $A - LC$  is Hurwitz and Metzler). In the present work, we are going to avoid such a restriction. First, we need the following assumptions.

**Assumption 1.**  $\|x\| \leq X$ ,  $\|u\| \leq U$  and  $\|y\| \leq Y$ , the constants  $X > 0$ ,  $U > 0$  and  $Y > 0$  are given.

Boundedness of the state  $x$  and the input  $u$  is a standard assumption in the estimation theory.

**Assumption 2.** Let  $\underline{x} \leq x \leq \bar{x}$  for some  $\underline{x} \in \mathbb{R}^n$  and  $\bar{x} \in \mathbb{R}^n$ , then  $\underline{f}(t, \underline{x}, \bar{x}, u) \leq f(t, x, u, \varrho) \leq \bar{f}(t, \underline{x}, \bar{x}, u)$  for some given  $\underline{f} : \mathbb{R}^{2n+m+1} \rightarrow \mathbb{R}^n$ ,  $\bar{f} : \mathbb{R}^{2n+m+1} \rightarrow \mathbb{R}^n$  and all  $t \geq 0$ ,  $\|u\| \leq U$ ,  $\varrho \in \Theta$ .

**Assumption 3.** There exist matrix functions  $L : \mathbb{R}^{p+m+1} \rightarrow \mathbb{R}^{n \times p}$ ,  $P : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ ,  $P(\cdot) = P(\cdot)^T \succ 0$  such that for all  $t \geq 0$  and  $\|u\| \leq U$ ,  $\|y\| \leq Y$ :

$$\begin{aligned}p_1 I_n &\preceq P(t) \preceq p_2 I_n, \quad p_1, p_2 > 0; \\ \dot{P}(t) + D(t, y, u)^T P(t) + P(t) D(t, y, u) + P(t)^2 + Q &\preceq 0, \\ D(t, y, u) &= A(t, y, u) - L(t, y, u) C(t, u), \quad Q = Q^T \succ 0.\end{aligned}$$

Assumption 2 states that if the bounds  $\underline{x}$ ,  $\bar{x}$  on the state value  $x$  are given, then the values of the nonlinear function  $f$  are enclosed in the interval  $[\underline{f}, \bar{f}]$  for all  $\varrho \in \Theta$  (for a continuous  $f$ , the computation of  $\underline{f}$ ,  $\bar{f}$  for given  $\underline{x}$ ,  $\bar{x}$  and a convex  $\Theta$  is a routine operation in the interval arithmetic [13]). In assumption 3 the observer gain  $L(t, y, u)$  is introduced, that ensures stability of the time-varying matrix  $D(t, y, u)$  with the Lyapunov function matrix  $P(t)$ , this assumption determines the generic stability conditions of the estimation dynamics. Due to assumption 1 the matrix  $A$  is varying in a compact domain, then Linear Parameter-Varying or polytopic system results [1], [2], [14], [9] can be used to compute a gain  $L$  satisfying assumption 3. If  $D(t, y, u) = D(t)$ , then this assumption is a conventional requirement for linear time-varying systems, if in addition  $D(t)$  is periodical, then this inequality can be solved as a differential equation [21].

Under these assumptions, if we additionally assume that the matrix  $D$  is Metzler, then the following interval observer can be designed [3], [12], [17]:

$$\begin{aligned}\dot{\underline{x}} &= A(t, y, u)\underline{x} + \underline{f}(t, \underline{x}, \bar{x}, u) + L(t, y, u)[y - C(t, u)\underline{x}], \\ \dot{\bar{x}} &= A(t, y, u)\bar{x} + \bar{f}(t, \underline{x}, \bar{x}, u) + L(t, y, u)[y - C(t, u)\bar{x}].\end{aligned}\tag{4}$$

**Theorem 1.** Let assumptions 1–3 hold, and the matrix  $D(t, y, u)$  be Metzler for all  $t \geq 0$  and  $\|u\| \leq U$ ,  $\|y\| \leq Y$ . Let one of the following conditions be satisfied:

- 1)  $|f(t, \underline{x}, \bar{x}, u)| < +\infty$ ,  $|\bar{f}(t, \underline{x}, \bar{x}, u)| < +\infty$  for any  $t \geq 0$ ,  $\|u\| \leq U$  and all  $\underline{x} \in \mathbb{R}^n$ ,  $\bar{x} \in \mathbb{R}^n$ ;
- 2) for any  $t \geq 0$ ,  $\|x\| \leq X$ ,  $\|u\| \leq U$ ,  $\varrho \in \Theta$  and all  $\underline{x} \in \mathbb{R}^n$ ,  $\bar{x} \in \mathbb{R}^n$

$$|f(t, x, u, \varrho) - \underline{f}(t, \underline{x}, \bar{x}, u)|^2 + |\bar{f}(t, \underline{x}, \bar{x}, u) - f(t, x, u, \varrho)|^2 \leq \beta|x - \underline{x}|^2 + \beta|\bar{x} - x|^2 + \alpha$$

for some  $\alpha \in \mathbb{R}_+$ ,  $\beta \in \mathbb{R}_+$ , and

$$\beta I_n - Q + R \preceq 0, R = R^T \succ 0.$$

Then in (3), (4) the variables  $\underline{x}(t)$  and  $\bar{x}(t)$  remain bounded for all  $t > 0$  and

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t),$$

provided that  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ .

*Proof:* Consider the interval estimation errors  $\bar{e} = \bar{x} - x$ ,  $\underline{e} = x - \underline{x}$ :

$$\begin{aligned} \dot{\underline{e}} &= D(t, y, u)\underline{e} + f(t, x, u, \varrho) - \underline{f}(t, \underline{x}, \bar{x}, u), \\ \dot{\bar{e}} &= D(t, y, u)\bar{e} + \bar{f}(t, \underline{x}, \bar{x}, u) - f(t, x, u, \varrho). \end{aligned}$$

Due to assumption 2 for a Metzler matrix  $D$ , for all  $t \geq 0$  the properties  $f[t, x(t), u(t), \varrho] \geq \underline{f}[t, \underline{x}(t), \bar{x}(t), u(t)]$ ,  $\bar{f}[t, \underline{x}(t), \bar{x}(t), u(t)] \geq f[t, x(t), u(t), \varrho]$  and

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t)$$

are satisfied, provided that  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ . To prove that the variables  $\underline{x}(t)$ ,  $\bar{x}(t)$  are bounded, consider the Lyapunov function  $V = \underline{e}^T P(t)\underline{e} + \bar{e}^T P(t)\bar{e}$  and evaluate its derivative:

$$\begin{aligned} \dot{V} &= \underline{e}^T [\dot{P}(t) + D(t, y, u)^T P(t) + P(t)D(t, y, u)]\underline{e} + \\ &\bar{e}^T [\dot{P}(t) + D(t, y, u)^T P(t) + P(t)D(t, y, u)]\bar{e} + \\ &2\underline{e}^T P(t)[f(t, x, u, \varrho) - \underline{f}(t, \underline{x}, \bar{x}, u)] + \\ &2\bar{e}^T P(t)[\bar{f}(t, \underline{x}, \bar{x}, u) - f(t, x, u, \varrho)]. \end{aligned}$$

Due to assumption 3 this equality can be rewritten as follows:

$$\begin{aligned} \dot{V} &\leq -\bar{e}^T Q\bar{e} - \underline{e}^T Q\underline{e} + \\ &|f(t, x, u, \varrho) - \underline{f}(t, \underline{x}, \bar{x}, u)|^2 + |\bar{f}(t, \underline{x}, \bar{x}, u) - f(t, x, u, \varrho)|^2. \end{aligned}$$

If the first condition of the theorem is true, then the terms  $|f(t, x, u, \varrho) - \underline{f}(t, \underline{x}, \bar{x}, u)|$  and  $|\bar{f}(t, \underline{x}, \bar{x}, u) - f(t, x, u, \varrho)|$  are bounded for any  $t \geq 0$ ,  $\|x\| \leq X$ ,  $\|u\| \leq U$ ,  $\varrho \in \Theta$  and all  $\underline{x} \in \mathbb{R}^n$ ,  $\bar{x} \in \mathbb{R}^n$ . Thus the errors  $\bar{e}$ ,  $\underline{e}$  are bounded by the standard Lyapunov arguments, and so are the variables  $\underline{x}$ ,  $\bar{x}$  (from assumption 1 the state  $x$  is bounded). If the second condition of the theorem holds, then this inequality becomes:

$$\dot{V} \leq -\bar{e}^T R\bar{e} - \underline{e}^T R\underline{e} + \alpha,$$

that implies boundedness of  $\underline{x}$ ,  $\bar{x}$  by the same arguments. ■

The result of Theorem 1 is based on the rather restrictive assumption that the matrix  $D$  is Metzler. All other

assumptions are rather common in the estimation theory (boundedness of the state  $x$  and the input  $u$  in assumption 1, existence of majorant functions for  $f$  from assumption 2, existence of the observer gain  $L$  with the corresponding Lyapunov matrix  $P$  in assumption 3, Lipschitz continuity or boundedness of  $\underline{f}$ ,  $\bar{f}$  stated in the theorem). For a constant matrix  $D$  this assumption is relaxed in Lemma 2, where it is shown that under conditions of assumption 3 (the matrix  $D$  is Hurwitz) there exists a static real similarity transformation matrix  $S$  with  $S^{-1}DS$  being Hurwitz and Metzler. In our case  $D(t, y, u)$  is a matrix variable, an extension of Lemma 2 for this case is developed below.

**Lemma 3.** *Let  $Z \in \Xi \subset \mathbb{R}^{n \times n}$  be a matrix variable satisfying the interval constraints  $\Xi = \{Z \in \mathbb{R}^{n \times n} : Z_a - \Delta \leq Z \leq Z_a + \Delta\}$  for some  $Z_a^T = Z_a \in \mathbb{R}^{n \times n}$  and  $\Delta \in \mathbb{R}_+^{n \times n}$ . If for some constant  $\mu \in \mathbb{R}$  and a diagonal matrix  $\Upsilon \in \mathbb{R}^{n \times n}$  the Metzler matrix  $R = \mu E_n - \Upsilon$  has the same eigenvalues as the matrix  $Z_a$ , then there is an orthogonal matrix  $S \in \mathbb{R}^{n \times n}$  such that the matrices  $S^T Z S$  are Metzler for all  $Z \in \Xi$  provided that  $\mu > n \|\Delta\|_{max}$ .*

*Proof:* Note that the matrices  $Z_a$  and  $R$  are symmetric by their definition. Thus there exist two orthogonal matrices  $O_Z \in \mathbb{R}^{n \times n}$ ,  $O_R \in \mathbb{R}^{n \times n}$  such that  $O_Z^T Z_a O_Z = O_R^T R O_R$  (these matrices may be composed by eigenvectors of  $Z_a$  and  $R$  respectively), the matrices  $O_Z$  and  $O_R$  can be chosen to satisfy  $\|O_Z\|_2 = \|O_R\|_2 = 1$ . Let  $S = O_Z O_R^T$  be another orthogonal matrix with  $\|S\|_2 = 1$ , then  $R = S^T Z_a S$ . For any  $Z \in \Xi$  we have  $S^T Z S = S^T (Z_a + \Pi) S$  for a matrix  $\Pi \in \mathbb{R}^{n \times n}$ ,  $-\Delta \leq \Pi \leq \Delta$ , then

$$S^T Z S = R + S^T \Pi S$$

and using (1) we get

$$\|S^T \Pi S\|_{max} \leq \|S^T \Pi S\|_2 = \|\Pi\|_2 \leq n \|\Pi\|_{max} \leq n \|\Delta\|_{max}.$$

All elements outside of the main diagonal in the matrix  $R$  equal  $\mu$ , thus all elements outside of the main diagonal of the matrix  $R + S^T \Pi S$  are not negative if  $\mu > n \|\Delta\|_{max}$ . ■

The matrix  $\mu E_n$  has one eigenvalue  $\mu n$  and the rest equal zero, the matrix  $R$  for  $\Upsilon = \rho I_n$  with  $\rho > \mu n$  is Hurwitz and Metzler. To apply this lemma assume that all its conditions are satisfied.

**Assumption 4.** *Let  $D(t, y, u) \in \Xi$  for all  $t \geq 0$ ,  $\|u\| \leq U$  and  $\|y\| \leq Y$ , where  $\Xi = \{D \in \mathbb{R}^{n \times n} : D_a - \Delta \leq D \leq D_a + \Delta\}$  for some  $D_a^T = D_a \in \mathbb{R}^{n \times n}$  and  $\Delta \in \mathbb{R}_+^{n \times n}$ . Let for some constant  $\mu > n \|\Delta\|_{max}$  and a diagonal matrix  $\Upsilon \in \mathbb{R}^{n \times n}$  the Metzler matrix  $R = \mu E_n - \Upsilon$  has the same eigenvalues as the matrix  $D_a$ .*

Under this assumption there is an orthogonal matrix  $S \in \mathbb{R}^{n \times n}$  such that the matrices  $S^T D(t, y, u) S$  are Metzler for all  $D(t, y, u) \in \Xi$ . Introduce new state variable  $z = S^T x$ , then the system (3) can be rewritten in the new coordinates:

$$\dot{z} = S^T A(t, y, u) S z + \phi(t, z, u, \varrho),$$

where  $\phi(t, z, u, \varrho) = S^T f(t, S z, u, \varrho)$ . Using (2) we have the following relations for  $x = S z$ :

$$\begin{aligned} \underline{x} &\leq x \leq \bar{x}, \\ \underline{x} &= S^+ \underline{z} - S^- \bar{z}, \quad \bar{x} = S^+ \bar{z} - S^- \underline{z}, \end{aligned} \tag{5}$$

where  $\underline{z} \leq z \leq \bar{z}$  are the interval estimates for the variable  $z$ . Under assumption 2 with the substitution of  $\underline{x}$ ,  $\bar{x}$  calculated in (5) we obtain:

$$\begin{aligned} \underline{\phi}(t, \underline{z}, \bar{z}, u) &= S^{+T} \underline{f}(t, \underline{x}, \bar{x}, u) - S^{-T} \bar{f}(t, \underline{x}, \bar{x}, u) \leq \\ \phi(t, z, u, \varrho) &\leq S^{+T} \bar{f}(t, \underline{x}, \bar{x}, u) - S^{-T} \underline{f}(t, \underline{x}, \bar{x}, u) = \bar{\phi}(t, \underline{z}, \bar{z}, u). \end{aligned}$$

In the new coordinates the interval observer takes form similar to (4):

$$\begin{aligned}\dot{\underline{z}} &= S^T A(t, y, u) S \underline{z} + \underline{\phi}(t, \underline{z}, \bar{z}, u) + \\ &\quad S^T L(t, y, u) [y - C(t, u) S \underline{z}], \\ \dot{\bar{z}} &= S^T A(t, y, u) S \bar{z} + \bar{\phi}(t, \underline{z}, \bar{z}, u) + \\ &\quad S^T L(t, y, u) [y - C(t, u) S \bar{z}].\end{aligned}\tag{6}$$

Now we are in position to prove the following relaxed variant of Theorem 1.

**Theorem 2.** *Let assumptions 1–4 hold. Let one of the following conditions be satisfied:*

- 1)  $|f(t, \underline{x}, \bar{x}, u)| < +\infty$  and  $|\bar{f}(t, \underline{x}, \bar{x}, u)| < +\infty$  for any  $t \geq 0$ ,  $\|u\| \leq U$  and all  $\underline{x} \in \mathbb{R}^n$ ,  $\bar{x} \in \mathbb{R}^n$ ;
- 2) for any  $t \geq 0$ ,  $\|x\| \leq X$ ,  $\|u\| \leq U$ ,  $\varrho \in \Theta$  and all  $\underline{z} \in \mathbb{R}^n$ ,  $\bar{z} \in \mathbb{R}^n$

$$\begin{aligned}|\phi(t, z, u, \varrho) - \underline{\phi}(t, \underline{z}, \bar{z}, u)|^2 + |\bar{\phi}(t, \underline{z}, \bar{z}, u) - \phi(t, z, u, \varrho)|^2 \leq \\ \beta |z - \underline{z}|^2 + \beta |\bar{z} - z|^2 + \alpha\end{aligned}$$

for some  $\alpha \in \mathbb{R}_+$ ,  $\beta \in \mathbb{R}_+$ , and

$$\beta I_n - S^T Q S + R \preceq 0, \quad R = R^T \succ 0.$$

Then in (3), (5), (6) the variables  $\underline{x}(t)$  and  $\bar{x}(t)$  are bounded for all  $t > 0$  and

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t),$$

provided that  $\underline{z}(0) \leq z(0) \leq \bar{z}(0)$ .

*Proof:* Consider the dynamics of the interval estimation errors  $\bar{e} = \bar{z} - z$ ,  $\underline{e} = z - \underline{z}$ :

$$\begin{aligned}\dot{\bar{e}} &= S^T D(t, y, u) S \bar{e} + \bar{\phi}(t, \underline{z}, \bar{z}, u) - \phi(t, z, u, \varrho), \\ \dot{\underline{e}} &= S^T D(t, y, u) S \underline{e} + \phi(t, z, u, \varrho) - \underline{\phi}(t, \underline{z}, \bar{z}, u).\end{aligned}$$

Due to assumption 2 for a Metzler matrix  $D$ , for all  $t \geq 0$  the properties  $\phi[t, z(t), u(t), \varrho] \geq \underline{\phi}[t, \underline{z}(t), \bar{z}(t), u(t)]$ ,  $\bar{\phi}[t, \underline{z}(t), \bar{z}(t), u(t)] \geq \phi[t, z(t), u(t), \varrho]$  and

$$\underline{z}(t) \leq z(t) \leq \bar{z}(t)$$

are satisfied, provided that  $\underline{z}(0) \leq z(0) \leq \bar{z}(0)$ . To prove that the variables  $\underline{x}(t)$ ,  $\bar{x}(t)$  are bounded, consider the Lyapunov function  $V = \underline{e}^T S^T P(t) S \underline{e} + \bar{e}^T S^T P(t) S \bar{e}$  derivative:

$$\begin{aligned}\dot{V} &= \underline{e}^T S^T [\dot{P}(t) + D(t, y, u)^T P(t) + P(t) D(t, y, u)] S \underline{e} + \\ &\quad \bar{e}^T S^T [\dot{P}(t) + D(t, y, u)^T P(t) + P(t) D(t, y, u)] S \bar{e} + \\ &\quad 2 \underline{e}^T S^T P(t) S [\phi(t, z, u, \varrho) - \underline{\phi}(t, \underline{z}, \bar{z}, u)] + \\ &\quad 2 \bar{e}^T S^T P(t) S [\bar{\phi}(t, \underline{z}, \bar{z}, u) - \phi(t, z, u, \varrho)].\end{aligned}$$

Due to assumption 3 this equality can be rewritten as follows:

$$\begin{aligned}\dot{V} \leq -\bar{e}^T S^T Q S \bar{e} - \underline{e}^T S^T Q S \underline{e} + \\ |\phi(t, z, u, \varrho) - \underline{\phi}(t, \underline{z}, \bar{z}, u)|^2 + |\bar{\phi}(t, \underline{z}, \bar{z}, u) - \phi(t, z, u, \varrho)|^2.\end{aligned}$$

If the first condition of the theorem is true, then the terms  $|\phi(t, z, u, \varrho) - \underline{\phi}(t, \underline{z}, \bar{z}, u)|$  and  $|\bar{\phi}(t, \underline{z}, \bar{z}, u) - \phi(t, z, u, \varrho)|$

are bounded for any  $t \geq 0$ ,  $\|x\| \leq X$ ,  $\|u\| \leq U$ ,  $\varrho \in \Theta$  and all  $z \in \mathbb{R}^n$ ,  $\bar{z} \in \mathbb{R}^n$ . Thus the errors  $\bar{e}$ ,  $e$  are bounded by the standard Lyapunov arguments, and so are the variables  $z$ ,  $\bar{z}$ . Due to (5) the same is true for the variables  $\underline{x}$ ,  $\bar{x}$ . If the second condition of the theorem holds, then this inequality takes form:

$$\dot{V} \leq -\bar{e}^T R \bar{e} - e^T R e + \alpha,$$

that implies boundedness of  $\underline{x}$ ,  $\bar{x}$  by the same arguments. ■

This theorem proposes an interval observer for an LTV (LPV) system explicitly skipping the requirement on cooperativity of the closed loop matrix  $D$ . Indeed according to assumption 3 it is only stable and there exists a transformation of coordinates that makes the closed loop dynamics cooperative. To apply this theorem, if we have a solution  $L$  from Assumption 3 ensuring the estimation error stability, then we may try to calculate  $D_a$  and  $\Delta$  for the obtained  $D$  in order to verify that the transformation matrix  $S$  exists. Next, the interval observer equations are given in (6).

#### IV. EXAMPLES

To illustrate the proposed results let us consider two examples.

##### A. Lorenz chaotic system

Consider a variant of the Lorenz model:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \quad y = x_1, \\ \dot{x}_2 &= -\eta x_2 - x_1 x_3 + \varrho x_1, \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2, \end{aligned}$$

where  $\eta$ ,  $\beta$ ,  $\sigma$  are constant parameters assumed to be known, parameter  $\varrho$  is time-varying and unknown,  $\varrho_{min} \leq \varrho \leq \varrho_{max}$  where  $\varrho_{max} > 0$  and  $\varrho_{min} > 0$  are given. The system can be represented in the form (3) using the designations:

$$A(y) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 0 & -\eta & -y \\ 0 & y & -\beta \end{bmatrix}, \quad f(y, \varrho) = \begin{bmatrix} 0 \\ \varrho y \\ 0 \end{bmatrix}, \\ C = [1 \ 0 \ 0].$$

We will assume that assumption 1 holds for the chosen parameters and initial conditions. Obviously the assumption 2 is satisfied for

$$\bar{f}(y) = \begin{cases} \varrho_{max} y & \text{if } y > 0 \\ \varrho_{min} y & \text{if } y \leq 0 \end{cases}, \quad \underline{f}(y) = \begin{cases} \varrho_{min} y & \text{if } y > 0 \\ \varrho_{max} y & \text{if } y \leq 0 \end{cases},$$

since  $y$  is bounded, these limiting functions are also bounded and the first condition of theorems 1 or 2 is verified. Direct computation shows that assumption 3 is true for  $L = [0 \ \sigma \ 0]^T$  and  $P = I_3$ . In assumption 4 we can choose

$$D_a = \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & -\eta & 0 \\ 0 & 0 & -\beta \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & y_{max} \\ 0 & y_{max} & 0 \end{bmatrix},$$

where  $y_{max}$  is the maximal admissible amplitude of the output.

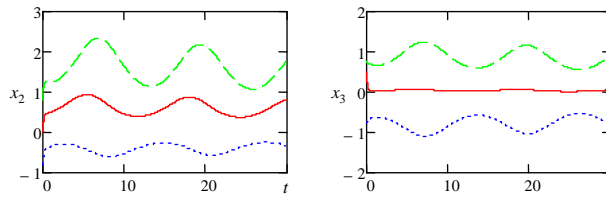


Figure 1. The results of simulation for Lorenz model for  $\varrho_{min} = 8$ ,  $\varrho_{max} = 12$

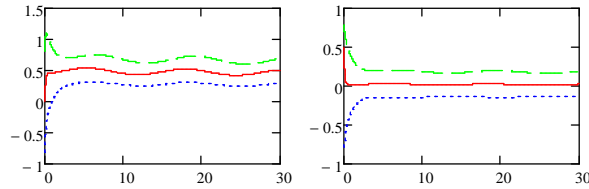


Figure 2. The results of simulation for Lorenz model for  $\varrho_{min} = 9.5$ ,  $\varrho_{max} = 10.5$

The following values of parameters are used for simulation:

$$\sigma = 1, \eta = 10, \beta = 10.$$

In addition two scenarios are considered:

$$\begin{aligned} \varrho_{min} = 8, \varrho_{max} = 12, \varrho(t) &= 10 + 2\sin(0.5t); \\ \varrho_{min} = 9.5, \varrho_{max} = 10.5, \varrho(t) &= 10 + 0.5\sin(0.5t). \end{aligned}$$

For these values  $y_{max} = 1$  and

$$\begin{aligned} R &= \mu E_n - \Upsilon, \mu = 3y_{max}, \Upsilon = 10I_3, \\ S &= \begin{bmatrix} 0.577 & 0.577 & 0.577 \\ -0.816 & 0.735 & 0.082 \\ 0 & -0.779 & 0.779 \end{bmatrix}. \end{aligned}$$

The results of interval estimation for the unmeasured coordinates  $x_2$  and  $x_3$  are shown in figures 1 and 2. As we can conclude, the width of the estimated interval is proportional to the uncertainty size, i.e.  $\varrho_{max} - \varrho_{min}$  in this example.

## B. LTV system

Now consider the LTV system

$$\dot{x} = A(t)x + b(t), y = x_2,$$

where  $x \in \mathbb{R}^2$  is the state and

$$A(t) = \begin{bmatrix} -0.632 - 0.8 \sin(t) & 0.5 \cos(3t) \\ -0.7 \cos(2t) & 0.3 \sin(t) \end{bmatrix},$$



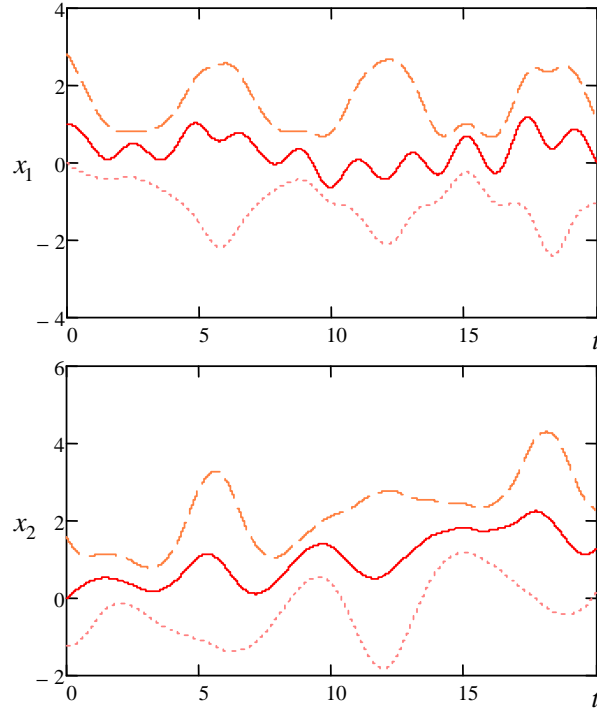


Figure 3. The results of simulation for LTV system

$b(t)$  is an uncertain input such that

$$\begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix} = \underline{b} \leq b(t) \leq \bar{b} = \begin{bmatrix} 0.3 \\ 0.6 \end{bmatrix},$$

during simulation  $b(t) = [0.1 + 0.2 \sin(0.5t) \quad 0.1 + 0.5 \cos(1.5t)]^T$  (assumption 2 holds). The matrix  $A(t)$  is unstable and not Metzler, but on a finite time interval assumption 1 is satisfied. For  $L = [0 \quad 4.368]^T$  the matrix  $D(t) = A(t) - LC$  admits the requirements of assumption 3 with a periodical  $0.01I_2 \leq P(t) \leq I_2$  and  $Q = 0.1I_2$ . Finally, the assumption 4 is satisfied for

$$D_a = \begin{bmatrix} -0.632 & 0 \\ 0 & -4.368 \end{bmatrix}, \Delta = \begin{bmatrix} 0.8 & 0.8 \\ 0.8 & 0.8 \end{bmatrix},$$

$$R = \begin{bmatrix} -2 & 1.8 \\ 1.8 & -3 \end{bmatrix}, S = \begin{bmatrix} 0.796 & -0.605 \\ 0.605 & 0.796 \end{bmatrix}.$$

Therefore, all conditions of Theorem 2 are valid and the results of simulation of the obtained interval observer are presented in Fig. 3 for both coordinates. In this example the estimated interval width is also predefined by the model uncertainty  $b(t)$ .

## V. CONCLUSION

The paper is devoted to an interval observer design for the LTV systems and the time-varying nonlinear systems in an output canonical form. It is the first time that the interval observer is designed for time-varying systems with a non-Metzler matrix  $D$ . A static transformation of coordinates is proposed mapping a stable LPV systems to another LPV system that is stable and cooperative. Thus the assumption that there exists an observer gain that

makes the estimation error dynamics stable and cooperative is relaxed. The observer gain has to ensure stability of the estimation error as usual, next a static transformation of coordinates is proposed, that provides the required cooperativity. The efficiency is shown on examples of computer simulation.

The relaxation of symmetry of the matrix  $D_a$  introduced in the conditions of the Lemma 3 and the stability conditions used in assumption 3 are the future directions of research.

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